Grade 6 Math Circles
Oct 11/12/13, 2022
Counting - Lesson

Today, we will learn how to count the number of different possible outcomes of specific actions and events. This type of mathematical counting is the foundation of an entire branch of math called **combinatorics**. Whenever we approach a counting problem, we need to understand:

- **What** are we counting: Numbers? Choices? Candy bars? What assumptions are we making?
- **How** are we counting: When do we use addition? Multiplication? There are formal rules but it's more important to understand the underlying logic. Tip: if you ever feel lost, solve the same problem with smaller numbers (where you can count without having to use fancy methods)

Counting with Addition and Subtraction

Rule of Sum:
Suppose there are m items of type A and n items of type B. The total number of items of type A **OR** type B is $m + n$. *Note: we are assuming that there are no items of both types*

Suppose we want to pick a movie to watch. Our possibilities include: 2 romance movies, 3 sci-fi movies, 4 thriller movies, and 2 movies of other genres. How many choices do we have? Intuitively, we add $2 + 3 + 4 + 2$ to get 11 as our answer. The above rule simply formalizes this process.

Exercise 1
Garry the Gardener has a garden with 19 tomatoes, 21 zucchinis, 27 cucumbers, and 13 carrots. How many vegetables does Garry have in his garden?

Exercise 1 Solution
Aside: Are tomatoes vegetables? Are cucumbers vegetables? For this problem, assume that both of these are vegetables.
Using the Rule of Sum, the answer is

$$19 + 21 + 27 + 13 = 80$$

Definition: If a and n are whole numbers, and a divides exactly into n (without remainder) then we say that n is **divisible** by a. Eg. 6 is divisible by 3 because $6 = 3 \times 2$ but 7 is not divisible by 3.

Example 1

How many whole numbers less than or equal to 100 are divisible by either 3 or 5?

Example 1 Solution - Wrong

The largest multiple of 3 which is at most 100 is $3 \times 33 = 99$, therefore there are 33 multiples of 3 less than or equal to 100 (these 33 multiples would be $3 \times 1, 3 \times 2, ..., 3 \times 33$).

The largest multiple of 5 which is at most 100 is $5 \times 20 = 100$, therefore there are 20 multiples of 5 less than or equal to 100.

Finally, we can use the Rule of Sum to find that there are $33 + 20 = 53$ numbers less than or equal to 100 which are divisible by either 3 or 5.

Example 1 Solution - Fixed

In the above solution, multiples of 15 are counted as both multiples of 5 and multiples of 3. That is, we first counted the multiples of 3:

$$3, 6, 9, 12, 15, 18, ..., 99$$

and then the multiples of 5:

$$5, 10, 15, 20, ..., 100$$

but 15 is in both lists despite the fact we only want to count it once. This is called **overcounting**. To obtain the correct answer, we need to subtract the multiples of 15 (i.e. the numbers we counted twice).

There are 6 multiples of 15 which are at most 100: 15, 30, 45, 60, 75, and 90. Therefore our final answer should be

$$53 - 6 = 47$$

We can generalize the Rule of Sum to include cases where 'items' might be both type A and type B:

Rule of Sum (generalized version):

Suppose there are m items of type A, n items of type B and k items that are both type A and type B. The total number of items of type A **OR** type B is $m + n - k$.

Complementary Counting

Sometimes it's easier to count the items we don't want rather than the ones we do want. For example, if we are picking 29 students from a class of 30 to go on a field trip, would it be easier to count all possible combinations of 29 students or count the number of ways we can pick 1 student to leave behind?

Exercise 2

How many positive whole numbers less than or equal to 1000 are not divisible by 6?

Exercise 2 Solution

Instead of counting the numbers which are not divisible by 6, we can count the numbers which are divisible by 6 and subtract that from 1000. The multiples of 6 which are less than 1000 are

$$1 \times 6,\ 2 \times 6,\ \ldots\ 165 \times 6,\ 166 \times 6$$

since $166 \times 6 = 996 < 1000 < 1002 = 167 \times 6$. Therefore, there are 166 numbers less than 1000 which are divisible by 6.

We can conclude that there are $1000 - 166 = 834$ numbers less than 1000 which are not divisible by 6.

Fundamental Counting Principle

Example 2

Suppose you are making an ice cream sundae. There are 4 flavours, 2 sauces, and 3 toppings that you can choose from. You are to select one of each. How many different sundaes could you make?

Example 2 Solution

Let S be a sauce and T a topping. How many sundaes are there with sauce S and topping T? There are 4: one for each flavour. In fact, for any sauce/topping combination we choose, there will be 4 possible sundaes (one for each flavour). Therefore

$$\#\text{ of sundaes} = 4\times\ (\#\text{ of ways of picking one sauce and one topping})$$

How many ways can we pick one sauce and one topping? Well for each sauce, there are 3 possible toppings. Since we have 2 sauces, there are 2×3 sauce/topping combinations. We can conclude

$$\# \text{ of sundaes} = 4 \times 2 \times 3 = 24$$

Notice

$$\# \text{ of sundaes} = (\# \text{ of flavours}) \times (\# \text{ of sauces}) \times (\# \text{ of toppings})$$

This leads us to another important rule in counting:

Fundamental Counting Principle:
If there are to m items of type A and n items of type B, then there are $m \times n$ ways of picking one item of type A **AND** one item of type B.

The Fundamental Counting Principle is used in a type of counting called **constructive counting**. This occurs when we construct a choice/item using smaller choices/items (eg. constructing an ice cream sundae using flavour, sauce, topping).

Exercise 3

How many 6-digit numbers are there in base 7? That is, how many 6-digit numbers do not use the digits 7, 8, or 9?

Exercise 3 Solution

Note leading 0's are not allowed so there are there are 6 choices for the first digit and 7 choices for each subsequent digit. Using the Fundamental Counting Principle, there are

$$6 \times 7 \times 7 \times 7 \times 7 \times 7 = 6 \times 7^5 = 100842$$

6-digit numbers in base 7.

Permutations

Often, we count different ways of choosing objects from a fixed group of objects. In this case, there are two things we should be aware of:

1. **order**: does the order of our choices matter?
2. **repetition**: are we allowed to pick the same object repeatedly?

When rearranging a list of numbers, objects or people, order matters and repetition is not allowed.

Example 3

How many ways can one rearrange the letters of the word MATH?

Example 3 Solution

A few possible rearrangements of the word MATH:

MATH, TAHM, THAM, HAMT, HTAM, ...

We will construct rearrangements of the word MATH by picking one letter at a time:

1. There are 4 possible letters to choose from for the first letter of our rearrangement: M, A, T, H.
2. There 3 possible letters to choose from for the second letter (since one letter was already used)
3. There are 2 possible letters to choose from for the third letter (two letters were already used)
4. There is only 1 possible letter remaining for the final letter

Using the fundamental principle of counting, there are $4 \times 3 \times 2 \times 1 = 24$ possible rearrangements of the letters in the word MATH.

How many 4-digit numbers use the digits 1, 2, 3, and 4 (each exactly once)? The problem is the exact same as Example 3: let $M = 1$, $A = 2$, $T = 3$, $H = 4$. In fact, the number of arrangements of any 4 different objects will be an equivalent problem to Example 3.

In general, how many ways can we order n **distinct** items? We can use the same logic as Example 3 to find the answer

$$n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$$

Factorials

Factorial notation is used to write this operation: define $0! = 1$ and

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

for $n > 0$. $n!$ is read as "n-factorial".

Examples:

1. $1! = 1$
2. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
3. $11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 39916800$

We can multiply and divide factorials just like any other numbers:

$$\frac{27!}{26!} = \frac{27 \times 26 \times ... \times 1}{26!} = \frac{27 \times 26!}{26!} = 27$$

Note each factorial can be found using the previous factorial: $n! = n \times (n-1)!$ for all $n > 0$.

Exercise 4
Evaluate each of the following expressions.

(a) $7! = 7 \times 6 \times 5! = 42 \times 120 = 5040$

(b) $\frac{2022!}{2021!} = \frac{2022 \times 2021!}{2021!} = 2022$

(c) $5! + \frac{5! \times 9!}{10!} = 120 + \frac{5! \times 9!}{10 \times 9!} = 120 + \frac{120}{10} = 132$

What if we had n options in total to choose from but we only needed to order k of them? For example, suppose we wanted to know how many different 4-digit long passwords use the whole numbers from 1 to 7, without repeating digits. We have 7 (n) options but only need to order 4 (k) of them.

Permutations are a way of counting in this type of situation when there is **no repetition**.

$${}_nP_k = \frac{n!}{(n-k)!}$$

This is read as "n permute k" and counts how many ways we can **order k objects** from a total of n objects.

Why does this formula hold? We can use similar logic as the rearrangement problem: consider n items and we wish to count the number of k-object orderings (no repetition allowed). Then we can construct a k-ordering one item at a time:

- There will be n items to choose from for the first item in our ordering.
- There will be $n-1$ items to choose from for the second item in our ordering.
- ...
- There will be $n-k+1$ remaining items to choose from for the last item in our ordering.

Thus, the number of k-orderings is

$$\begin{aligned}{}_nP_k &= n \times (n-1) \times ... \times (n-k+1) \\ &= \frac{n \times (n-1) \times ... \times (n-k+1) \times (n-k) \times (n-k-1) \times ... \times 1}{(n-k) \times (n-k-1) \times ... \times 1} \\ &= \frac{n!}{(n-k)!}\end{aligned}$$

We can use the formula to find that there are

$${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 7 \times 6 \times 5 \times 4 = 42 \times 20 = 840$$

possible 4-digit long passwords that use whole numbers from 1 to 7, without repeating digits.

Exercise 5

Math Spheres is running a lottery. 10 balls, numbered 1 through 10, are placed in a bin. One at a time, four balls are drawn, and their numbers are recorded (in order) to form a winning list of four numbers. If the balls are not replaced after being drawn, how many possible winning combinations are there?

Exercise 5 Solution

We are looking for the number of 4-ball combinations (out of 10 total balls) in which order matters and repetition is not allowed. This is just 10 permute 4 so our answer is

$$_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!} = 10 \times 9 \times 8 \times 7 = 5040$$

Combinations

What if we had n objects in total and needed to choose k with **no repetition** (just like in a permutation) but now **order does not matter**?

Combinations are a way of counting in this type of situation when there is no repetition and **order doesn't matter**.

$$_nC_k = \frac{n!}{k!(n-k)!}$$

This is read as "n choose k" and counts how many ways we can choose k objects from a total of n objects.

Each permutation of k objects is obtained by first choosing k objects (without order) and then arranging them. There are $_nC_k$ ways of choosing k objects without order and $k!$ ways of arranging these k items. Thus,

$$_nP_k = \ _nC_k \times k!$$

which allows us to find the above formula.

Let's look at the formula for ${}_nC_k$ more closely:

- There is symmetry in the formula:

$$ {}_nC_k = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = {}_nC_{n-k} $$

  This makes sense since by choosing k items, we are also choosing $n-k$ items to exclude.

- When $k=0$, there is only one way to choose nothing:

$$ {}_nC_0 = \frac{n!}{0!n!} = \frac{1}{0!} = 1 $$

- When $k=1$, there are n ways of choosing one item from n items:

$$ {}_nC_1 = \frac{n!}{1!(n-1)!} = n $$

A popular alternative to writing ${}_nC_k$ is using the binomial coefficient:

$$ \binom{n}{k} := {}_nC_k = \frac{n!}{k!(n-k)!} $$

Example 4

(a) How many ways are there to form a 3-person committee from a party of 25 politicians?

(b) There are 12 people attending an event. If everyone shakes hands, how many handshakes will occur?

(c) How many 10-digit binary strings contain exactly six 1's?

Example 4 Solution

(a) This is the number of ways of choosing 3 people out of 25 people:

$$ {}_{25}C_3 = \frac{25!}{22! \times 3!} = \frac{25 \times 24 \times 23}{6} = 2300 $$

(b) The number of handshakes is equal to the number of pairs of people (since each pair of people shake hands exactly once). This is equal to the number ways of choosing 2 out of 12 people:

$$_{12}C_2 = \frac{12!}{10! \times 2!} = \frac{12 \times 11}{2} = 66$$

(c) There must be a 1 in the first position so this is equal to the number of ways of choosing which of the 9 remaining digits will be the 5 remaining 1's:

$$_{9}C_5 = \frac{9!}{5! \times 4!} = \frac{9 \times 8 \times 7 \times 6}{24} = 126$$